United Kingdom
Mathematics Trust

Intermediate Mathematical Challenge

Solutions 2020

For reasons of space, these solutions are necessarily brief.

There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation:

www.ukmt.org.uk

1. A The value of $2 - (3 - 4) - (5 - 6 - 7) = 2 - (-1) - (-1 - 7) = 2 + 1 - (-8) = 2 + 1 + 8 = 11$.

2. E The correct answer is a multiple of both 3 and 8, as $24 = 3 \times 8$. Although 100 is a multiple of 4, it is not a multiple of 8. Therefore even multiples of 100 are multiples of 8, but odd multiples are not. This leaves 200, 400 and 600, but of these only 600 is also a multiple of 3.
Hence the only option which is a multiple of 24 is 600.

3. B The required difference is 25% of (£37 - £17) = 25% of £20.
Hence the difference between 25% of £37 and 25% of £17 is £5.

4. D We choose units so the outer square has side-length 4, and hence area 16.
The unshaded area of the square consists of two congruent triangles of base 3 and height 2.
So the area of each unshaded triangle is $\frac{1}{2} \times 3 \times 2 = 3$.
Hence the area of the shaded part of the square is $16 - 2 \times 3 = 10$.
Therefore the fraction of the square which is shaded is $\frac{10}{16} = \frac{5}{8}$.

5. D First note that $AB = \sqrt{13}$, $AC = \sqrt{26}$, $AD = \sqrt{17}$ and $AE = \sqrt{13}$. Thus $AB = AE$.
Note also that $BC = \sqrt{13}$ and $CE = \sqrt{13}$. Therefore $ABCE$ is a rhombus.
Finally, $BE = \sqrt{26}$, so $ABCE$ is a rhombus with two equal diagonals and hence is a square.
So (3,5) is the odd point out.

6. C The values of the five options are 64, 243, 256, 125, 36 respectively.
Of these 256 is the largest, so 4^4 is the largest of the options.

7. B We label the five squares, from left to right, P, Q, R, S, T respectively.
In order to paint two adjacent squares, Lucy could paint P and Q, or Q and R, or R and S, or S and T. So in four of the finished grids, Lucy's red squares are adjacent to each other.

8. E Since the units digits of all the options are different, it is sufficient to add the units digits of the four squares in the sum. As $7^2 = 49$, 17^2 has a units digit of 9. In a similar way, we calculate that 19^2 has a units digit of 1, 23^2 has a units digit of 9 and 29^2 has a units digit of 1.
Therefore the units digit of the given sum is the units digit of $9 + 1 + 9 + 1$, that is 0.
As we are told that the correct answer is one of the options, we may deduc that it is 2020.

9. E Note that the numbers 123 to 213 are in both option A and option C. Also, numbers 213 to

231 are in both option B and option C, and numbers 231 to 312 are in both C and D. However, numbers 313 to 321 are in option E only. So that must be where Adam's house is.

10. **B** The value of $\dfrac{2468 \times 2468}{2468 + 2468}$ is $\dfrac{2468 \times 2468}{2 \times 2468} = \dfrac{2468}{2}$.

Hence the correct answer is 1234.

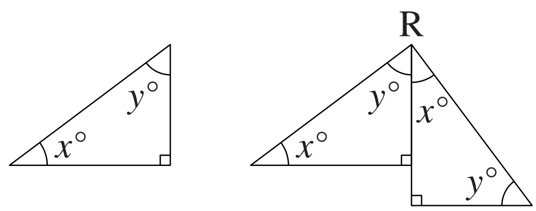
11. **E** There is exactly one route from "1" to "2", but two routes from "1" to "3" - one route directly from "1" to "3" and one route which moves from "1" to "2" to "3".
To move from "1" to "4", it is necessary to move from "2" directly to "4" or to move from "3" to "4". So the number of routes from "1" to "4" is $1 + 2 = 3$.
By a similar method, the number of routes from "1" to "5" is $2 + 3 = 5$ and from "1" to "6" is $3 + 5 = 8$. Finally, the number of routes from "1" to "7" is $5 + 8 = 13$.
(*Note that the numbers of routes to each of the squares is a term in the Fibonacci sequence.*)

12. **B** Let the number of chickens and goats sold be c and g respectively.
Then, since 80 animals are sold, we have $c + g = 80$.
Also, $2c + 4g = 200$, since the chickens have two legs and the goats have four legs.
Dividing the second equation by 2, we obtain $c + 2g = 100$.
Then, subtracting the first equation gives $g = 100 - 80 = 20$.
So 20 goats were sold.

13. **C** Half of 1.6×10^6 equals $0.8 \times 10^6 = 0.8 \times 10 \times 10^5 = 8 \times 10^5$.

14. **A** Note that $9 \times 11 \times 13 \times 15 \times 17$ is a multiple of 9. Therefore the sum of its digits is also a multiple of 9. The sum of the digits of '3n8185' equals $25 + n$. So $n = 2$, since 27 is a multiple of 9.

15. **A** The diagrams show one of the twenty-five congruent right-angled triangles and the two such triangles at vertex R.
Let the two acute angles in each of the triangles be $x°$ and $y°$.



Note that $x + y + 90 = 180$, as the interior angles of a triangle sum to 180°.
So $x + y = 90$. In the second diagram, the two angles which meet at R are $x°$ and $y°$, so we can deduce that angle QRP is a right angle.
Let the length of the hypotenuse of each small triangle be a cm.
Note that angle QRP is a right angle, PR has length $4a$ cm and RQ has length $3a$ cm.
So the lengths of the sides in triangle PQR are in the ratio 3 : 4 : 5.
Therefore the length of PQ, in cm, is $5 \times \frac{2.4}{4} = 3$.

16. **E** The sum of $\dfrac{1}{9} + \dfrac{1}{11}$ is $\dfrac{11}{99} + \dfrac{9}{99}$, that is $\dfrac{20}{99}$.

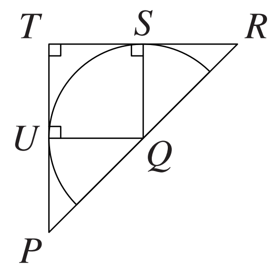
Now 20 and 99 do not have any factors in common except 1. So the fraction cannot be simplified.
In option A, we may write 0.10 as the fraction $\frac{10}{100}$, whose denominator is a power of 10. The same is true of options B, C and D. Therefore, when simplified, none of these fractions can have 99 as a denominator.
It is left to the reader to confirm, by division, that $\dfrac{20}{99} = 0.\dot{2}\dot{0}$.
An alternative argument follows.
Let $x = 0.\dot{2}\dot{0} = 0.202020....$ Then $100x = 20.202020...$.
Subtracting the first equation from the second gives $99x = 20$.

So $0.\dot{2}\dot{0} = \dfrac{20}{99}$.

17. **E** Suppose that at a particular stage there are m tarts available for a Knave to eat and that there are n left after he has finished eating.
Then $n = m - (\frac{1}{2}m + \frac{1}{2}) = \frac{1}{2}m - \frac{1}{2}$.
Therefore, $m = 2n + 1$.
As the Knave of Spades received one tart, then the number of tarts which the Knave of Clubs was given was $2 \times 1 + 1 = 3$.
Similarly, the number of tarts which the Knave of Diamonds was given was $2 \times 3 + 1 = 7$.
Finally, the number of tarts which the Knave of Hearts stole was $2 \times 7 + 1 = 15$.

18. **C** Let the length of each equal side of the given triangle be x.
Then, by Pythagoras' theorem, $x^2 + x^2 = y^2$. So $x^2 = \frac{y^2}{2}$.
In the triangle, four of the squares are shaded while the unshaded area consists of two squares and four half-squares.
Therefore, half of the area of the triangle is shaded.
Now the area of the triangle is $\frac{1}{2} \times x \times x = \frac{1}{2}x^2$.
Therefore the total shaded area of the triangle is $\frac{1}{4}x^2 = \frac{1}{4} \times \frac{y^2}{2} = \frac{y^2}{8}$.

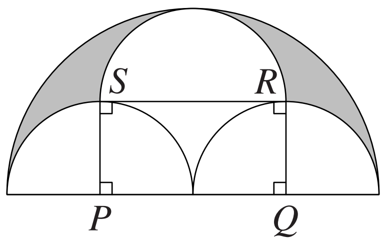
19. **B** The diagram shows the top left-hand corner of the original diagram.
The centre of the semicircle shown is Q. Also, U and S are the points where the edges of the bigger square touch the semicircle shown.
Therefore both QU and QS are radii of the semicircle and $\angle TSQ = \angle TUQ = 90°$. Also $\angle UTS$ is a right angle as it is the corner of a square. Therefore $UQST$ is a square. Hence $QS = ST$.



Note that $TR = TP$ because P and R are midpoints of the original large square. Therefore $\angle PRT = 45°$. So $QS = RS$.
Hence QS is half the length of TR, which is itself half of the length of a side of the outer square, which is 48 cm.
So the radius of the semicircle is one quarter of 48 cm = 12 cm.

20. **D** The first four expressions expand to $x^2 - 1$; $x^2 - \frac{1}{4}$; $x^2 - \frac{1}{9}$; $x^2 - \frac{1}{16}$ respectively.
Note that $\frac{1}{16} < \frac{1}{9} < \frac{1}{4} < 1$. Therefore the least value is $x^2 - \frac{1}{16}$, that is $(x + \frac{1}{4})(x - \frac{1}{4})$.
This result is irrespective of the value of x.

21. **B** In the diagram, SR is the diameter of the upper small semicircle and P and Q are the centres of the two lower small semicircles.
Note that the line SR touches the two semicircles with centres P and Q at points S and R respectively.
So $\angle SRQ = \angle RSP = 90°$.
Also, $SR = PQ = 2$ cm. Therefore $PQRS$ is a rectangle.



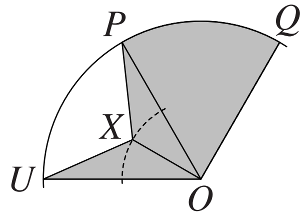
The total unshaded area in the diagram is the rectangle plus a semicircle and two quarter circles, that is, the rectangle plus a circle. So, in cm^2, it is $1 \times 2 + \pi \times 1^2 = 2 + \pi$.
So the total shaded area, in cm^2, is $\frac{1}{2} \times \pi \times 2^2 - (2 + \pi) = 2\pi - (\pi + 2) = \pi - 2$.
Hence the required area, in cm^2, is $\pi - 2$.

22. **C** The exterior angle of a regular pentagon is $\dfrac{360°}{5} = 72°$.
Therefore the interior angle of a regular pentagon, in degrees, is $180 - 72 = 108$. The angles at a point sum to 360°, so the reflex angle in the irregular quadrilateral, in degrees, is $360 - 108 = 252$. Finally the interior angles of a quadrilateral sum to 360°, so the sum of the marked angles, in degrees, is $360 - 252 = 108$.
(Note that the sum of the three marked angles equals the interior angle of the pentagon.)

23. **A** Let the small sides of each triangle have length r. This is also the side of the original square.
The longer side of each triangle is $\sqrt{2}r$, since that is the diagonal of the square.
Hence the perimeters of shapes P, Q, R are $(2 + 3\sqrt{2})r$, $(6 + \sqrt{2})r$ and $(4 + 3\sqrt{2})r$ respectively.
Now $6 + \sqrt{2} - (2 + 3\sqrt{2}) = 4 - 2\sqrt{2} = 2(2 - \sqrt{2})$, which is greater than zero.
Therefore the perimeter of P is less than the perimeter of Q.
Also, $4 + 3\sqrt{2} - (6 + \sqrt{2}) = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$, which is also greater than 0.
Therefore the perimeter of Q is less than the perimeter of R.
Hence, in ascending order, the lengths of the perimeters are P, Q, R.

24. **A** We are given that $10 \times 2^m = 2^n + 2^{n+2}$. Therefore $5 \times 2 \times 2^m = 2^n(1 + 2^2)$.
Hence $5 \times 2^{m+1} = 2^n \times 5$. So $m + 1 = n$. Therefore the difference between m and n is 1.

25. **A** The diagram shows exactly one third of the shaded area in the original diagram. It is made up of the quadrilateral $UOPX$, together with a sector of the outer circle, POQ, where O is the centre of the original circle.



Since P and Q are two of the six points equally spaced around the circle, $\angle POQ = \frac{1}{6} \times 360° = 60°$. The outer circle has radius 2 cm, so the area, in cm^2, of sector POQ is $\frac{1}{6} \times \pi \times 2^2 = \frac{2\pi}{3}$.

Since U and P are also equally spaced, $\angle UOP = 60°$. So $\angle UOX = 30°$.
Hence the area of triangle UOX, in cm^2, is $\frac{1}{2} \times 1 \times 2 \times \sin 30° = \frac{1}{2}$.
So the area of quadrilateral $UOPX$ is $1\ \text{cm}^2$.
Therefore the total shaded area, in cm^2, in the original diagram is $3 \times (\frac{2\pi}{3} + 1) = 2\pi + 3$.